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Modules Which Are Lifting Relative To Module Classes

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Abstract. In this paper, we study a module which is lifting and supplemented relative to a module class. Let R be a ring, and let $\mathcal{X}$ be a class of R-modules. We will define $\mathcal{X}$-lifting modules and $\mathcal{X}$-supplemented modules. Several properties of these modules are proved. We also obtain results for the case of specific classes of modules.

1. Introduction

Throughout this work all rings will be associative with identity and modules will be unital right modules.

Let R be a ring and let M be a right $R-$module. We will write $N \leq M$ to mean N is a submodule of the module M. A submodule N of M is said to be a *small* in M, denoted by $N \ll M$, whenever $L \leq M$ and $M = N + L$ then $M = L$. A module M is said to be *small* if M is small in $E(M)$, the injective hull of M. Given any submodule N of M, by a *supplement* of N in M we mean a submodule K of M, which is minimal in the collection of submodules L with the property $N + H = M$(see[10] and [12]). A nonzero module M is called *hollow* if every proper submodule of M is small in M. The module M is said to be a *lifting module* (or D_1-module) if for any submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$. On the other hand, an R-module M is said to be a *supplemented* (*weakly supplemented*) if every submodule of M has a supplement in M (see[10]). The module M is said to be an *amply supplemented* if for any two submodules N and K with $M = N + K$, N has a supplement in K(see[12]).

By a class $\mathcal{X}$ of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphisms, i.e., any module isomorphic to some module in $\mathcal{X}$ also belongs to $\mathcal{X}$. By an *$\mathcal{X}$-module* we mean any member of $\mathcal{X}$, and

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a submodule N of a module M is called *$\mathcal{X}$-submodule* of M if N is a $\mathcal{X}$-module. Motivated by the notion of lifting modules and supplemented modules we introduce here the concept of $\mathcal{X}$-lifting and $\mathcal{X}$-supplemented modules. Lifting modules are worthy of study in module theory since they are dual of extending modules, and there has been a great deal of work on lifting modules by many authors. As a generalization of lifting modules supplemented, weakly supplemented and amply supplemented modules are also studied in [8], [10] and [12]. Generalizations of supplemented and semiperfect modules with respect to a torsion theory have been considered in the literature (see for instance [7] and [13]). A similar approach (with more general classes than torsion theories) had been considered in [2]. Extending modules relative to module classes have been studied in [4], [5] and [6] and also in [14]. In this paper, we investigate these modules relative to a class $\mathcal{X}$ of modules.

Therefore we define *$\mathcal{X}$-lifting*, *$\mathcal{X}$-supplemented* and *$\mathcal{X}$-amply supplemented modules.* Various general properties of such modules are given. An $\mathcal{X}$-submodule N of M is called *$\mathcal{X}$-supplement* if N is a supplement of some submodule in M. An $R-$module M is said to be a *$\mathcal{X}$-supplemented* module if every $\mathcal{X}$-submodule of M has a supplement in M. The module M is said to be a *$\mathcal{X}$-lifting module* if for every $\mathcal{X}$-submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$. A module M is called a *$\mathcal{X}$-amply supplemented* if every $\mathcal{X}$-submodule N of M such that $M = N + K$, K contains a supplement of N in M. Let $\mathcal{M}$ denote the class of all R-modules. Then a module M is lifting (supplemented or amply supplemented) if and only if M is $\mathcal{M}$-lifting ($\mathcal{M}$-supplemented or $\mathcal{M}$-amply supplemented). For a class $\mathcal{X}$ of R-modules, a module M is $\mathcal{X}$-lifting if and only if every $\mathcal{X}$-submodule N of M has a decomposition $N = K \oplus L$, where K is a direct summand of M and $L \ll M$. Every $\mathcal{X}$-supplement submodule of M is $\mathcal{X}$-coclosed, and if M is $\mathcal{X}$-amply supplemented, then then every $\mathcal{X}$-coclosed submodule is $\mathcal{X}$-supplement submodule of M.

$\mathcal{X}_1.\mathcal{X}_2.\cdots\mathcal{X}_n$, $\oplus_{i=1}^{n}\mathcal{X}_i$ and $\mathcal{X}_1 : \mathcal{X}_2$ from old ones and study lifting property with respect to these classes. In particular, we use $\mathcal{H}$ for the class of all hollow R-modules, $f\mathcal{H}$ for all $R-$modules with finite hollow dimension, $\mathcal{S}$ for the semisimple R-modules and $\mathcal{F}$ for all finitely generated $R-$modules. Among others we prove, under some restrictions that M is $\mathcal{H}$-lifting if and only if M is $f\mathcal{H}$-lifting. And M is $\mathcal{S}$- and $\mathcal{F}$-lifting if and only if $\mathcal{S} : \mathcal{F}$-lifting.

2. The results

The following lemma is clear from definitions.

Lemma 2.1. *Let $\mathcal{X}$ be any class of R-modules. Then every $\mathcal{X}$-lifting module is $\mathcal{X}$-supplemented and $\mathcal{X}$-amply supplemented.*

There are $\mathcal{X}$-supplemented modules, which are not $\mathcal{X}$-lifting.

Example 2.2. Let Z denote the ring of integers and consider the Z-module $M = N \oplus (U/V)$, where $N = Z/8Z$ and the submodules $U = 2Z/8Z$ and $V = 4Z/8Z$

of N. Let $\overline{\overline{0}}$ and $\overline{\overline{2}}$ denote the elements of U/V. Let $\mathcal{X} = \{X \in Mod - Z : X2 \neq 0\} \cup \{0\}$, and $N_1 = (\overline{1}, \overline{\overline{2}})Z$, $N_2 = (\overline{2}, \overline{\overline{0}})Z$, $N_3 = (\overline{2}, \overline{\overline{2}})Z$, $N_4 = (\overline{1}, \overline{\overline{0}})Z$, $N_5 = (\overline{4}, \overline{\overline{0}})Z$ and $N_6 = (\overline{4}, \overline{\overline{2}})Z$. Then N_1, N_2, N_3 and N_4 are $\mathcal{X}$-submodules, N_1 and N_4 are direct summands of M and $N_2 \ll M$, $M = N_1 + N_3$, $N_5 = N_1 \cap N_3$, $N_5 \ll M$ and $M = N_1 \oplus N_6$. It follows that M is $\mathcal{X}$-supplemented. It is easily checked that N_3 is neither small in M nor has any nonzero submodule which is direct summand of M. Hence M is not $\mathcal{X}$-lifting.

Lemma 2.3. *Let M be a module. We consider the following for a class $\mathcal{X}$:*

(1) *M is $\mathcal{X}$-lifting if and only if every $\mathcal{X}$-submodule N of M has a decomposition $N = K \oplus L$, where K is a direct summand of M and L is small in M.*

(2) *Every direct summand of a $\mathcal{X}$-lifting module is $\mathcal{X}$-lifting.*

(3) *M is $\mathcal{X}$-lifting module if and only if every $\mathcal{X}$-supplement submodule of M is direct summand and M is $\mathcal{X}$-supplemented*

Proof. (1) Let N be a $\mathcal{X}$-submodule of M. If M is $\mathcal{X}$-lifting then N contains a direct summand K of M such that $M = K \oplus K'$ and $N \cap K'$ is small in K'. So $N = K \oplus (N \cap K')$ and $N \cap K'$ is small in M. Conversely, let N be a $\mathcal{X}$-submodule of M. By assumption, $N = K \oplus L$, where $M = K \oplus K'$ and $L \ll M$. Then $N \cap K' \cong L$, and since L is small in M and K' is direct summand, $N \cap K'$ is small in K'.
(2) Let $M = M_1 \oplus M_2$ be a $\mathcal{X}$-lifting module and N a $\mathcal{X}$-submodule of M_1. Then N is $\mathcal{X}$-submodule of M, and so there exists a direct summand K of M such that $K \leq N$ and $M = K \oplus L$ and $N \cap L \ll L$. Then $M_1 = K \oplus (M_1 \cap L)$ and $N \cap L \ll M_1 \cap L$.
(3) Let N be a $\mathcal{X}$-supplement of a submodule U of M. Then $M = N + U$ and $N \cap U \ll N$. By (1), $N = K \oplus L$, where $M = K \oplus K'$ and $L \ll M$. Hence $M = K \oplus U$ and $N = K \oplus (N \cap U)$. This implies that $N = K$. The rest is clear. □

Example 2.4.

(i) Let $\mathcal{X}$ be the class of all torsion Z-modules. The zero submodule of Z is the only $\mathcal{X}$-submodule of Z. Hence the Z-module Z is a $\mathcal{X}$-lifting module.

(ii) Let $\mathcal{X}$ be the class of all torsion free Z-modules. The zero submodule is the only small submodule of Z, and for any non-zero submodules N and K with $N + K = Z$, $N \cap K$ is not a small submodule of Z and so the Z-module Z is not $\mathcal{X}$-lifting module.

(iii) Let $\mathcal{X}$ denote the class of all finitely generated Z- modules. Since every $\mathcal{X}$-submodule of Q and Q/Z is small, Q and Q/Z are $\mathcal{X}$-lifting modules.

(iv) Let $\mathcal{X}$ be the class of all torsion free Z-modules and p any prime integer and $M = (Z/pZ) \oplus Z$. It is clear that from (ii) and Lemma 2.3, the Z-module M is not $\mathcal{X}$-lifting.

(v) Let R be a ring and $\mathcal{X}$ denote the class of all injective R-modules. Then every R-module M is $\mathcal{X}$-lifting. Let N be a $\mathcal{X}$-submodule of the module M. Then N is injective R-submodule of M and so N is a direct summand of M.

(vi) Let $\mathcal{P}$ denote the class of projective modules and assume that any injective module is $\mathcal{P}$-lifting. Then every projective is the direct sum of an injective and a small module. Those types of decompositions occur in the study of H-rings (Harada-Rings)(see[11]).

(vii) Let R be a ring and $\mathcal{F}$ denote the class of all finitely generated $R-$modules. $\mathcal{F}$-supplemented modules are called *f-supplemented* in the literature. We take the polynomial ring $C[x]$ in one variables over complex numbers C. The localization $R = C[x]_{(x)}$ of $C[x]$ by the maximal ideal generated by x is a discrete valuation ring. Let $M = C[[x]]$ be the power series ring over C. Then M is an R-module such that $M = R + xM$ and thus $x(M/R) = M/R$. If N were a supplement of R in M, then $M = R + N$ implied $M/R \cong N/(N \cap R)$. Hence $x(N/(N \cap R)) = N/(N \cap R)$ and thus $N = xN + (N \cap R) = xN$ as $N \cap R$ is small in N. This implies $N = \cap_{k=1}^{\infty} x^k N \subseteq \cap_{k=1}^{\infty} x^k M = 0$. Hence $R = M = C[[x]]$, which is incorrect. Therefore the cyclic module R has no supplement in the R-module $M = C[[x]]$, i.e., M is not f-supplemented, although R is local (semiperfect).

Remark: It is actually possible to show that over a commutative noetherian ring a finitely generated module has a supplement in any module extension if and only if it is linearly compact. Hence every module is f-supplemented if and only if every finitely generated module has a supplement in any module extension of it.

Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of modules. We write $\mathcal{X} \leq \mathcal{Y}$ in case every object of $\mathcal{X}$ is in $\mathcal{Y}$. The next result is clear.

Lemma 2.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of modules with $\mathcal{X} \leq \mathcal{Y}$. Every $\mathcal{Y}$-lifting (or $\mathcal{Y}$-supplemented) module is $\mathcal{X}$-lifting (or $\mathcal{Y}$-supplemented).*

Example 2.6. Let $\mathcal{X} = \{X \in Mod - Z : X2 = 0\}$ and $\mathcal{Y} = \{Y \in Mod - Z : Y4 = 0\}$ and let M be the $Z-$module $(Z/2Z) \oplus (Z/8Z)$. Then $\mathcal{X} \leq \mathcal{Y}$ and M is $\mathcal{X}$-lifting but not an $\mathcal{Y}$-lifting module.

Proof. The submodules $N_1 = (\bar{1}, \bar{0})Z$, $N_2 = (\bar{0}, \bar{4})Z$ and $N_3 = N_1 \oplus N_2$ are $\mathcal{X}$-submodules of M. Let $N_4 = (\bar{0}, \bar{1})Z$. Then $M = N_3 + N_4 = N_1 \oplus N_4$, $N_1 \leq N_3$ and $N_3 \cap N_4 = N_2 \ll N_4$, and since N_1 is direct summand, we have M is $\mathcal{X}$-lifting. Let $N = (\bar{1}, \bar{2})Z$. Then N is an $\mathcal{Y}-$submodule of M and it does not contain any submodule as a direct summand of M. Hence M is not an $\mathcal{Y}$-lifting module (see also [8]). $\square$

Let n be a positive integer and let $\mathcal{X}_i (1 \leq i \leq n)$ be classes of R-modules. Classes of R-modules can be combined in different ways to give other classes and we examine how lifting and supplemented properties behave under these constructions. Then $\oplus_{i=1}^{n} \mathcal{X}_i$ is defined to be the class of R-modules M such that $M = \oplus_{i=1}^{n} M_i$

is direct sum of $\mathcal{X}_i$-submodules M_i $(1 \leq i \leq n)$. In particular, if $\mathcal{X}_i = \mathcal{X}$ we shall denote the class $\oplus_{i=1}^{n}\mathcal{X}_i$ by $\mathcal{X}^{\oplus}$. Then for each i with $1 \leq i \leq n$, $\mathcal{X}_i \leq \oplus_{i=1}^{n}\mathcal{X}_i$.

Theorem 2.7. *With the above notation, an R-module M is $(\oplus_{i=1}^{n}\mathcal{X}_i)$-lifting if and only if M is $\mathcal{X}_i$-lifting for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 2.5. To prove the converse we can suppose by induction that $n = 2$. Assume that M is $\mathcal{X}_1$ and $\mathcal{X}_2$- lifting. Let N be a $\mathcal{X}_1 \oplus \mathcal{X}_2$-submodule of M. Then there exist a $\mathcal{X}_1$-submodule N_1 and a $\mathcal{X}_2$-submodule N_2 of M such that $N = N_1 \oplus N_2$. By assumption, there exist direct summands K_1 and K_2 of M with $K_1 \leq N_1$ and $K_2 \leq N_2$ such that $M = K_1 \oplus K_1'$ and $M = K_2 \oplus K_2'$ with $N_1 \cap K_1' \ll K_1'$ and $N_2 \cap K_2' \ll K_2'$. Notice that N_1 is isomorphic to $(N_1 \oplus K_2) \cap K_2'$. Then $(N_1 \oplus K_2) \cap K_2'$ is a $\mathcal{X}_1$-module. By hypothesis, there exists $A \leq (N_1 \oplus K_2) \cap K_2'$ such that $M = A \oplus B$ and $((N_1 \oplus K_2) \cap K_2') \cap B \ll B$ for some $B \leq M$. Then $N_1 \oplus K_2 = A \oplus ((N_1 \oplus K_2) \cap B)$, $K_2' = A \oplus (K_2' \cap B)$ and $A \oplus K_2 = A \oplus ((A \oplus K_2) \cap B)$. Hence

$$N_1 \oplus K_2 = A + K_2 + ((N_1 + K_2) \cap K_2') = ((A \oplus K_2) \cap B) + ((N_1 \oplus K_2) \cap K_2').$$

We intersect $N_1 \oplus K_2$ by B to obtain

$$(N_1 \oplus K_2) \cap B = ((A \oplus K_2) \cap B) + ((N_1 \oplus K_2) \cap K_2' \cap B) \leq A \oplus K_2 \oplus (K_2' \cap B).$$

It follows that $M = N_1 + K_2 + K_2' = A + (N_1 \oplus K_2) \cap B + (K_2' \cap B) = A \oplus K_2 \oplus (K_2' \cap B)$. Hence $A \oplus K_2$ is a direct summand of M. Since $M = A + K_2 + K_2'$ and $A \oplus K_2 \leq N_1 \oplus K_2$ and $N_2 = K_2 \oplus (N_2 \cap K_2')$, $N_1 \oplus K_2 = A \oplus K_2 \oplus ((N_1 \oplus K_2) \cap K_2' \cap B)$. Hence $N = N_1 \oplus K_2 \oplus (N_2 \cap K_2') = A \oplus K_2 \oplus ((N_1 \oplus K_2) \cap K_2' \cap B) \oplus (N_2 \cap K_2')$, where $A \oplus K_2$ is a direct summand of M and $((N_1 + K_2) \cap K_2' \cap B) \oplus (N_2 \cap K_2')$ is small in M. By Lemma 2.3, M is $\mathcal{X}_1 \oplus \mathcal{X}_2-$lifting. □

Corollary 2.8. *Let $\mathcal{X}$ be a class of $R-$modules. A module M is $\mathcal{X}^{\oplus}$-lifting if and only if M is $\mathcal{X}$-lifting.*

The following example is as an illustration of Theorem 2.7.

Example 2.9. Let M denote the $Z-$module $(Z/2Z) \oplus (Z/8Z) \oplus (Z/3Z)$. Let $\mathcal{X}_1 = \{X \in Mod - Z : X2 = 0\}$, $\mathcal{X}_2 = \{X \in Mod - Z : X3 = 0\}$ and $\mathcal{Y} = \{Y \in Mod - Z : Y4 = 0\}$. Then it is easily seen that M is $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_1 \oplus \mathcal{X}_2$-lifting module. It has observed in Example 2.6 that the Z-module $(Z/2Z) \oplus (Z/8Z)$, which is a direct summand of M is not $\mathcal{Y}$-lifting. Hence M is not $\mathcal{Y}$-lifting.

Let M be a module and $\{N_\lambda\}_{\lambda \in \Lambda}$ a family of submodules of M. The family $\{N_\lambda\}_{\lambda \in \Lambda}$ is called *coindependent* if for any $\lambda \in \Lambda$ and any finite subset $F \subseteq \Lambda - \{\lambda\}$, $M = N_\lambda + \cap_{\beta \in F} N_\beta$ with the convention, that the intersection with an empty index set is set to be M.

For any ring R, $\mathcal{H}$ will denote the class of hollow $R-$modules and $f\mathcal{H}$ will denote the class of all $R-$modules with finite hollow dimension. Recall that $f\mathcal{H}$ consists of all $R-$modules M, which do not contain an infinite coindependent family of submodules, equivalently M contains a finite coindependent family of submodules $\{N_1, N_2, \cdots, N_n\}$ such that $\cap_{i=1}^n N_i$ is small in M and M/N_i is hollow module for every $1 \le i \le n$. Note that $\mathcal{H}$ and $f\mathcal{H}$ are closed under isomorphisms, factor modules and direct summands and $\mathcal{H} \subseteq f\mathcal{H}$. Note that a module M is hollow if and only if every coindependent family of submodules has exactly one element(See[9]).

Lemma 2.10. *Let M be an $\mathcal{H}$-lifting module. Then every hollow submodule of M is small or a direct summand of M.*

Proof. Let M be an $\mathcal{H}$-lifting module and N a hollow submodule of M. Then N is an $\mathcal{H}$-submodule and so there exists $A \le N$ such that $M = A \oplus B$ with $N \cap B \ll B$. Hence $N = A$ or $N = N \cap B$. $\square$

Proposition 2.11. *Let M be a module of which submodules having finite hollow dimension are amply supplemented. Then any submodule of finite hollow dimension is either small in M or contains a hollow submodule which is non-small in M.*

Proof. Let N be any submodule of finite hollow dimension in the module M. We may assume by induction that N has hollow dimension two so that N has submodules N_1 and N_2 such that $N = N_1 + N_2$ and N/N_i is hollow for $i = 1, 2$. By hypothesis there exists $L_2 \le N_2$ such that $N = N_1 + L_2$ with $N_1 \cap L_2 << L_2$. It is easily seen that L_2 is hollow module. If L_2 is not small in M, we are done. Assume that $L_2 \ll M$. By hypothesis there exists $L_1 \le N_1$ such that $N = L_1 + L_2$ with $L_1 \cap L_2 \ll L_1$. Then L_1 is hollow module. If L_1 is not small in M, then we are done. Otherwise $N = L_1 + L_2$ is small in M as required. $\square$

Theorem 2.12. *Let M be a module of which submodules having finite hollow dimension are amply supplemented. Then M is $\mathcal{H}$-lifting if and only if M is $f\mathcal{H}$-lifting.*

Proof. Since $\mathcal{H} \subseteq f\mathcal{H}$ sufficiency follows by Lemma 2.5. Assume that M is $\mathcal{H}$-lifting and let N be any $f\mathcal{H}$-submodule of M. If N is small in M, we are done. Assume that N is not a small submodule of M. Let n denote the hollow dimension of N. Then N contains a finite coindependent family of submodules $\{N_1, N_2, ..., N_n\}$ such that $\cap_{i=1}^n N_i$ is small in N and N/N_i is hollow module for every $1 \le i \le n$ (see [9 ,3.1.2, page 30]). By Lemma 2.10 and Proposition 2.11, there exists a hollow submodule L_1 of N such that $M = L_1 \oplus L_1'$. If $N \cap L_1' \ll L_1'$, there is nothing to prove. So assume that $N \cap L_1'$ is not small in L_1'. Then $N \cap L_1'$ is not small in M and it has finite hollow dimension. By the same reasoning, $N \cap L_1'$ contains a direct summand L_2 of $M = L_2 \oplus L_2'$ so that $M = L_1 \oplus L_2 \oplus (L_1' \cap L_2')$ and $N = L_1 \oplus L_2 \oplus (N \cap (L_1' \cap L_2'))$. If $N \cap (L_1' \cap L_2')$ is not small in L_2', we repeat the same procedure so that $N \cap (L_1' \cap L_2')$ contains a direct summand L_3, we have $M = L_3 \oplus L_3'$, and then $M = L_1 \oplus L_2 \oplus L_3 \oplus (L_1 \cap L_2 \cap L_3)$ and $N = L_1 \oplus L_2 \oplus L_3 \oplus (N \cap (L_1' \cap L_2' \cap L_3'))$. If $(N \cap (L_1' \cap L_2' \cap L_3')$ is not small in M, we proceed on this way so that $M = \oplus_{i=1}^k L_i \oplus M_k$ and $N = \oplus_{i=1}^{k+1} L_i \oplus N_k$, where $M_k = \cap_{i=1}^k L_i'$ and $N_k = N \cap M_k$. Hence we obtain a sequence of submodules $N \ge N_1 \ge N_2 \ge N_3 \ge ....$ Since N has finite hollow dimension, there exists j

such that for every $t \geq j$, N_j/N_t is small in N/N_t. By construction of submodules N_k(k =1,2,$\cdots$), N_j/N_t is isomorphic to a direct summand of N/N_t. This is a contradiction being N of finite hollow dimension. Hence there exists a finite integer t such that $N = (\oplus_{i=1}^{t} L_i) \oplus N_t$, where $\oplus_{i=1}^{t} L_i$ is a direct summand of M and N_t is small in L_t' and in M. This completes the proof by Lemma 2.3. □

Let $\mathcal{X}_i (1 \leq i \leq n)$ be classes of R-modules. Following [4], $\mathcal{X}_1.\mathcal{X}_2.\cdots\mathcal{X}_n$ will denote the class of R-modules M such that there exists a chain of submodules $0 = N_0 \leq N_1 \leq \cdots \leq N_n = M$ such that N_i/N_{i-1} is a $\mathcal{X}_i$-module $(1 \leq i \leq n)$, and if N_{i-1} is a small submodule of M then N_i is a $\mathcal{X}_i$-module $(2 \leq i \leq n)$. Note that $\mathcal{X}_i \leq \mathcal{X}_1.\mathcal{X}_2.\cdots.\mathcal{X}_n$ for all $1 \leq i \leq n$. On the other hand $\mathcal{X}_1 : \mathcal{X}_2$ will denote the class of R-modules M such that there exists a submodule N of M such that N is a $\mathcal{X}_1$-module and M/N is a $\mathcal{X}_2$-module. Then $\mathcal{X}_1 \leq \mathcal{X}_1 : \mathcal{X}_2$ and $\mathcal{X}_2 \leq \mathcal{X}_1 : \mathcal{X}_2$.

Theorem 2.13. *Let $\mathcal{X}_i (1 \leq i \leq n)$ be classes of R-modules. Then an $R-$module M is $\mathcal{X}_i$-lifting for every $1 \leq i \leq n$ if and only if M is $\mathcal{X}_1.\mathcal{X}_2.\cdots\mathcal{X}_n$-lifting.*

Proof. The sufficiency follows by Lemma 2.5. Conversely, we may assume by induction that $n = 2$. Assume that M is $\mathcal{X}_i$-lifting for i = 1,2 and let N be a $\mathcal{X}_1.\mathcal{X}_2$-submodule of M. Then there exists a nonzero submodule N_1 such that $N_1 \in \mathcal{X}_1$ and $N/N_1 \in \mathcal{X}_2$. Since M is $\mathcal{X}_1$-lifting, there exists $A_1 \leq N_1$ and $B_1 \leq M$ such that $M = A_1 \oplus B_1$ and $N_1 \cap B_1 \ll B_1$. Then $N_1 = A_1 \oplus (N_1 \cap B_1)$ and $N = A_1 \oplus (N \cap B_1)$. Since $(N \cap B_1)/(N_1 \cap B_1) \cong N/N_1 \in \mathcal{X}_2$ and $N_1 \cap B_1$ is a small submodule of M, by hypothesis, $N \cap B_1 \in \mathcal{X}_2$. By assumption, there exist $A_2 \leq N \cap B_1$ and $B_2 \leq M$ such that $M = A_2 \oplus B_2$ and $(N \cap B_1) \cap B_2 \ll B_2$. Then $B_1 = A_2 \oplus (B_1 \cap B_2), M = A_1 \oplus A_2 \oplus (B_1 \cap B_2)$ and $N \cap B_1 \cap B_2 \ll M$. Hence $N = A_1 \oplus A_2 \oplus (N \cap B_1 \cap B_2)$. By Lemma 2.3, M is a $\mathcal{X}_1.\mathcal{X}_2$-lifting module. □

Lemma 2.14. *Let M be a module and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be classes of R-modules. If M is $\mathcal{X}_1 : \mathcal{X}_2$-lifting, then M is $\mathcal{X}_i$-lifting for $i = 1,2$.*

Proof. Clear from Lemma 2.5. □

Example 2.15 shows that the converse of Lemma 2.14 is not true in general.

Example 2.15. There are modules classes $\mathcal{X}_1$ and $\mathcal{X}_2$ and a module M such that M is both $\mathcal{X}_1$ and $\mathcal{X}_2$-lifting but not $\mathcal{X}_1 : \mathcal{X}_2$-lifting.

Proof. Let p be any prime integer and $\mathcal{X}_1 = \mathcal{X}_2 = \{T \in Mod - Z : Tp = 0\}$ and $M = (Z/pZ) \oplus (Z/p^3Z)$. Let $M_1 = (\overline{1}, \overline{0})Z$, $N = (\overline{1}, \overline{p})Z$, $N_1 = (\overline{0}, \overline{p^2})Z$, $N = M_1 \oplus N_1$. Then M_1, N_1 and N_2 are all $\mathcal{X}_1$ and $\mathcal{X}_2$ submodules of M, M_1 is a direct summand and N_1 is small in M. By Lemma 2.3, M is both $\mathcal{X}_1$ and $\mathcal{X}_2$-lifting module. Also N_1 is an $\mathcal{X}_1$-module and N/N_1 is an $\mathcal{X}_2$-module. Hence N is an $\mathcal{X}_1 : \mathcal{X}_2$-submodule of M. It is easy to check that N is neither small nor a direct summand nor contains any direct summand of M. Hence M is not a $\mathcal{X}_1 : \mathcal{X}_2$-lifting module. □

Theorem 2.16. *Let $\mathcal{S}$ denote the class of all semisimple R-modules and $\mathcal{F}$ the class of all finitely generated R-modules. Then a module M is both $\mathcal{S}$ and $\mathcal{F}$-lifting* *if and only if M is $\mathcal{S} : \mathcal{F}$-lifting.*

Proof. One way is clear by Lemma 2.14. Assume that the module M is $\mathcal{S}$-lifting and $\mathcal{F}$-lifting. Let N be any $\mathcal{S} : \mathcal{F}$-submodule of M. Then there exists $N_1 \leq N$ such that $N_1 \in \mathcal{S}$ and $N/N_1 \in \mathcal{F}$. By assumption, there exists $A \leq N_1$ such that $M = A \oplus B$ and $N_1 \cap B \ll B$. Then $N/N_1 \cong (N \cap B)/(N_1 \cap B)$. Since N/N_1 is an $\mathcal{F}$-module, there exists $C \leq N \cap B$ such that C is an $\mathcal{F}$-module and $N \cap B = C + (N_1 \cap B)$. Since $N_1 \cap B$ is semisimple, $N \cap B = C \oplus L$ for some $L \leq N_1 \cap B$. Since C is $\mathcal{F}$-module, there exists $U \leq C$ such that $M = U \oplus V$ and $C \cap V \ll V$ for some $V \leq M$. Hence $M = A \oplus U \oplus (B \cap V)$ and $N = A \oplus U \oplus ((C \cap V) \oplus L)$. Since $A \oplus U$ is a direct summand of M and $(C \cap V) \oplus L$ is small in M, this completes the proof by Lemma 2.3. □

Proposition 2.17. *Let $\mathcal{F}$ be the class of all finitely generated R-modules and M an indecomposable R-module. Then M is $\mathcal{F}$-lifting if and only if $RadM = M$ or M is local module.*

Proof. Sufficiency is clear. For the necessity assume that M is a $\mathcal{F}$-lifting module. Let $x \in M$. Then there exist $A \leq xR$ and $B \leq M$ such that $M = A \oplus B$ and $(xR) \cap B \ll B$. By hypothesis, two cases arise $M = A$ or $A = 0$. First case implies $M = xR$ and so $RadM$ is small in M. Hence M is local. In the second case $M = B$. Then xR is small in M. Assume that M is not local module. Then for each $x \in M$ the second case will occur and xR will be small in M. It follows that $RadM = M$. □

Can we characterize $\mathcal{X}$-lifting modules via objects of the class $\mathcal{X}$? For this question, $T_{\mathcal{X}}(M)$ will denote *trace* of $\mathcal{X}$ in M, i.e., the sum of $\mathcal{X}$-submodules of M.

Lemma 2.18. *Let $\mathcal{X}$ be any class of R-modules. Then $T_{\mathcal{X}}(M) = \Sigma\{T_{\mathcal{X}}(N) : N$ is a $\mathcal{X}$-submodule of $M\}$.*

Proof. Clear. □

Lemma 2.19. *Assume that $\mathcal{X}$ is closed under homomorphic images and $f : M \longrightarrow N$ is a homomorphism from a module M to a module N. Then $f(T_{\mathcal{X}}(M)) \leq T_{\mathcal{X}}(N)$.*

Proof. Because if A is a $\mathcal{X}$-submodule of M, then $f(A)$ is a $\mathcal{X}$-submodule of N. □

Corollary 2.20. *Assume that $\mathcal{X}$ is closed under direct sums and homomorphic images. Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of modules M_i for all $i \in I$. Then $T_{\mathcal{X}}(M) = \oplus_{i \in I} T_{\mathcal{X}}(M_i)$.*

Proof. For each $i \in I$, $\pi_i : M \longrightarrow M_i$ be the canonical projection. By Lemma 2.19, $\pi_i(T_{\mathcal{X}}(M)) \leq T_{\mathcal{X}}(M_i)$ for all $i \in I$. Hence $T_{\mathcal{X}}(M) \subseteq \oplus_{i \in I} T_{\mathcal{X}}(M_i)$. Conversely, $\oplus_{i \in I} T_{\mathcal{X}}(M_i) \subseteq T_{\mathcal{X}}(M)$ by Lemma 2.18. □

Obviously, if M does not contain any non-zero $\mathcal{X}$-submodule, i.e. $T_{\mathcal{X}}(M) = 0$, then M is trivialy $\mathcal{X}$-lifting.

Example 2.21. Let $\mathcal{X}$ be the class of all torsion Z-modules and M be the Z-module Z. Since the zero submodule of Z is the only $\mathcal{X}$-submodule of M, i.e., $T_{\mathcal{X}}(M) = 0$, M is $\mathcal{X}$-lifting (compare with Example 2.4 (i)).

Remark. What if $\mathcal{X}$ is closed under direct sums and homomorphic images, then $T_{\mathcal{X}}(M)$ belongs to $\mathcal{X}$. Hence, we should be characterize $\mathcal{X}$-lifting module M via $T_{\mathcal{X}}(M)$.

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